Emergence of Hierarchy in Networked Endorsement Dynamics

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Overview

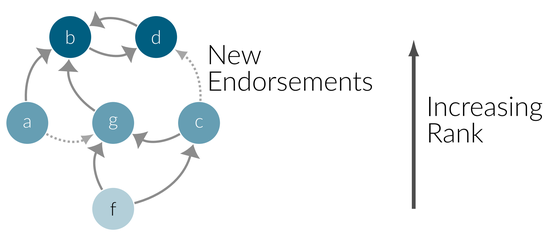
Many social and biological systems are **hierarchical**:

- Social ranks in animal groups.
- Perceived quality of universities and departments.
- Tiers of political candidates; athletes; movie stars…

We introduce a tractable, **coevolving network model of hierarchy**. This model…

- …possesses distinct egalitarian and hierarchical regimes, separated by a **phase transition**.
- …allows **tractable inference** from data.
- …highlights inferred **timescales** and **prestige preferences** in real-world systems!

Model Definition: Prestige-Driven Endorsements

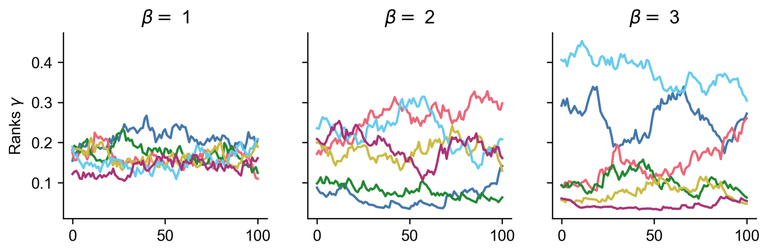


A first-order Markov chain model with state $\mathbf{A} \in \mathbb{R}^{n \times n}$. In each time step:

- Compute **ranks** $\boldsymbol{\gamma}_\beta \in \mathbb{R}^n$ from $\mathbf{A}$. We use SoftMax SpringRank [2] with inverse temperature β (prestige preference).
- Uniformly random agent I endorses agent $J \sim \text{Categorical}(\boldsymbol{\gamma}_\beta)$.
- **Update**: for memory parameter $\lambda \in \mathbb{R}$,

$$\mathbf{A}' = \lambda \mathbf{A} + (1 - \lambda)\mathbf{E}_{IJ} \; .$$

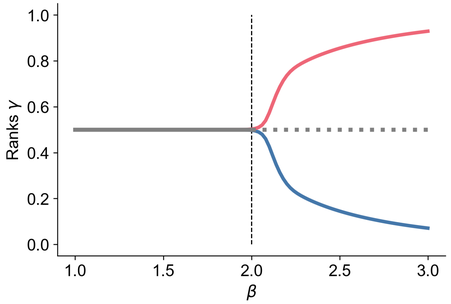
Model Dynamics Depend on β



Larger $\beta \to$ more stratified ranks.

Phase Transition in the Long-Memory Limit

Theorem: The *egalitarian regime* $\boldsymbol{\gamma}_\beta = n^{-1}\mathbf{e}$ is linearly stable in expectation as $\lambda \to 1$ if and only if $\beta < 2$.



Maximum-Likelihood Inference

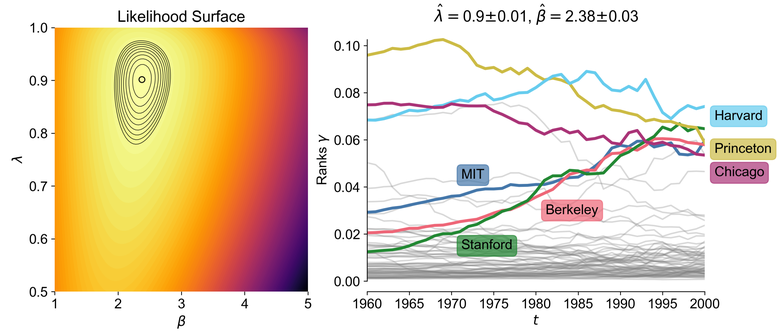
We can **estimate** the prestige-sensitivity β and memory λ from empirical data via maximum-likelihood. Let $\mathbf{k} = \mathbf{e}^T \boldsymbol{\Delta}^{(t)}$. The log-likelihood is

$$\mathcal{L}(\lambda, \beta; \{\mathbf{A}^{(t)}\}) = \beta \sum_{i=1}^{n} k_i s_i - (\mathbf{k}^T \mathbf{e}) \log \left(\sum_{i=1}^{n} e^{\beta s_i} \right) + C \; .$$

We maximize this expression with respect to λ and β. We estimate uncertainty by inverting the observed Fisher information matrix.

Example: US Mathematics PhD Exchange

University I endorses J when I hires a PhD graduate from J [3, 1].



We observe **long memory** (large λ) and **strong hierarchy** (large β).

Discussion

We have developed a novel model of **emergent hierarchy** in social networks.

- Model does not assume fixed "fitness" or "quality" for agents.
- Model gives interpretable time-dependent importance (centrality) scores with **direct dynamical interpretations**.

Future directions include analysis of the non-egalitarian regime and more complex generating mechanisms.

References

[1] The Mathematics Genealogy Project. `https://www.genealogy.math.ndsu.nodak.edu/index.php`.

[2] Caterina De Bacco, Daniel B Larremore, and Cristopher Moore. A physical model for efficient ranking in networks. *Science Advances*, 4(7):eaar8260, 2018.

[3] Dane Taylor, Sean A Myers, Aaron Clauset, Mason A Porter, and Peter J Mucha. Eigenvector-based centrality measures for temporal networks. *Multiscale Modeling & Simulation*, 15(1):537–574, 2017.